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Mutual-Chern-Simons effective theory of doped antiferromagnets

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A mutual-Chern-Simons Lagrangian is derived as a minimal field theory description of the phase-string model for doped antiferromagnets. Such an effective Lagrangian is shown to retain the full symmetries of parity, time-reversal, and global SU(2) spin rotation, in contrast to conventional Chern-Simons theories where first two symmetries are usually broken. Two ordered phases, i.e., antiferromagnetic and superconducting states, are found at low temperatures as characterized by "dual" Meissner effects and dual flux quantization conditions due to the mutual-Chern-Simons gauge structure. A "dual" confinement in charge/spin degrees of freedom occurs such that no true spin-charge separation is present in these ordered phases, but the spin-charge separation/deconfinement serves as a driving force in the unconventional phase transitions of these ordered states to disordered states.

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I. INTRODUCTION

Gauge theory description has become essential in studying doped Mott insulators. The physical necessity may be traced to the Hilbert space restriction in a doped Mott insulator. For instance, the high-T_c cuprate superconductors at half-filling are believed to be an antiferromagnetic (AF) Mott insulator[1], in which the charge sector at low energy is totally frozen up by the Coulomb interaction. After doping, the low-energy charge degrees of freedom do emerge, but remain highly restricted in the Hilbert space[1]. To characterize such a Hilbert space restriction, a spin-charge separation description, namely, by introducing[2,3,4] spinless "holon" of charge $+e$ and neutral spin-1/2 "spinon" as the essential *building blocks* of the restricted Hilbert space, has become an effective and useful way. Here "holons" and "spinons" do not necessarily turn out to be true low-lying elementary excitations in the end, because generally local gauge field(s) will emerge[5,6] to mediate interactions between these "holons" and "spinons", and may even lead to the *confinement* of them if either a true spin-charge separation does not exist or the decomposition is not done in a correct way. In general, one always ends up with a gauge theory description for doped Mott insulators where the gauge interaction can greatly influence the low-energy dynamics of the charge and spin degrees of freedom.

Several kinds of (2+1)-dimensional gauge theories have been proposed for doped two-dimensional (2D) spin-1/2 antiferromagnets related to the high-T_c cuprates. A U(1) gauge theory[6,7] based on the slave-boson approach to the $t-J$ model is one of the most intensively studied. Its gauge structure may be directly visualized by noting the gauge invariance of the electron operator in the slave-boson decomposition[2]

$$c_{i\sigma} = b_i^\dagger f_{i\sigma} \tag{1}$$

under a U(1) transformation: $b_i \to e^{i\theta_i} b_i$ and $f_{i\sigma} \to e^{i\theta_i} f_{i\sigma}$, where b_i denotes the bosonic "holon" operator and $f_{i\sigma}$ the fermionic "spinon" operator. Along the same line, the SU(2) non-Abelian gauge theories[8] and Z_2 gauge theories[9] have also been proposed and studied.

The slave-boson approach is considered to be convenient in dealing with the superconducting (SC) regime but has less advantage in describing the AF state near half-filling. On the other hand, gauge theories[10,11,12,13,14,15] based on the slave-fermion, Schwinger boson decomposition are believed to be useful in studying a lightly doped AF state. Here the electron operator is written as[16]

$$c_{i\sigma} = f_i^\dagger b_{i\sigma} \tag{2}$$

where f_i denotes the fermionic "holon" operator and $b_{i\sigma}$ the bosonic "spinon" operator. Besides the slave-boson and slave-fermion decompositions, slave-anyon decompositions have also been investigated[17,18,19,20]. Different gauge structures mentioned above originate from different decompositions and/or different mean-field decouplings. But a common feature for these gauge theories is that both "holon" and "spinon" share the same gauge field.

Recently, a different gauge theory description has been constructed[21] based on a distinctive decomposition of the electron operator[22,23]

$$c_{i\sigma} = h_i^\dagger b_{i\sigma} e^{i\hat{\Theta}_{i\sigma}} \tag{3}$$

which is known as the *bosonization*[22] or *phase string decomposition*[23] because holon and spinon operators, $h_i^\dagger$ and $b_{i\sigma}$, are both bosonic, with the fermionic commutations relations of the electron operator being restored by the phase string operator, $e^{i\hat{\Theta}_{i\sigma}} = (-\sigma)^i e^{i\frac{1}{2}\left[\Phi_i^b - \sigma\Phi_i^h\right]}$. Here *internal* gauge invariance appears as U(1)×U(1): $h_i \rightarrow e^{i\phi_i} h_i$ and $\Phi_i^b \rightarrow \Phi_i^b + 2\phi_i$; $b_{i\sigma} \rightarrow e^{i\sigma\chi_i} b_{i\sigma}$ and $\Phi_i^h \rightarrow \Phi_i^h + \chi_i$. Consequently there exist a pair of U(1)×U(1) gauge fields coupling to the holon and spinon fields, respectively, in the resulting gauge theory, called the phase string model, derived[21] based on the decomposition (3) and the bosonic resonating-valence-bond (RVB) mean-field saddle-point, where the normal U(1) gauge freedom[24] (like the one in the slave-boson case) is broken by the mean-field decoupling[25].

In the slave-boson (or slave-fermion) U(1) gauge theory, the external U(1) gauge field (i.e., the electromagnetic field) couples to *both* holons and spinons[6,7], thanks to the same *internal* U(1) gauge field they share. So both holon and spinon carry some fractions of the electron charge[7,26]. In contrast, in the phase string model, the external electromagnetic field only couples to the holon degrees of freedom, without being directly transferred to the spinon part as the latter sees a different gauge field. In this sense, the holon carries the full charge of $+e$ in the phase string model.

Without a bare kinetic energy, the single U(1) gauge field in the slave-boson (or slave-fermion) theory fluctuations strongly[7,13], which makes the theory a strong-coupling one. On the other hand, the U(1)×U(1) gauge fields are topological ones with their strengths constrained to the densities of two matter fields (see Sec. II) such that their fluctuations are much more mildly, suitable for a perturbative treatment. In particular, the no-double-occupancy constraint of the doped Mott insulator, which is enforced by the violent gauge fluctuations in the slave-boson (or slave-fermion) theory, is realized in the phase string model in a quite different way. Namely, the U(1)×U(1) topological gauge fields will introduce mutual repulsions between holons and spinons, where holons perceive spinons as vortices and vice versa. As it is well known, a particle cannot go to the core of a vortex of its own field where the density of such a matter field vanishes. In the phase string model such a vortex core of one species is always occupied by a different species such that the no-double-occupancy is naturally enforced.

Furthermore, the weak (logarithmic) confinement of spinons and holons at low energies and low temperatures has been also found[27,28,29] in the phase string model, as opposed to the strong confinement in usual 2D compact U(1) gauge models in slave-boson or slave-fermion theory[30,31]. In the latter, an effective gauge theory may have a serious infrared divergence[6,7] which makes the gauge theory very difficult to deal with mathematically. The former is usually much more manageable than the latter in this regard.

However, the Hamiltonian formalism[21] of the phase string model, in which a gauge field seen by one species is constrained to the density (number) of different species, is not very convenient for studies beyond the mean-field level. In this paper, we shall develop a Lagrangian (path-integral) formalism of the phase string model. We show that the effective low-energy Lagrangian describes two matter fields, holon and spinon, minimally couple to two *different* U(1) gauge fields. These gauge fields have no their own kinetic terms either, but there is a *mutual-Chern-Simons* term which entangles two gauge fields together. We call this as a mutual-Chern-Simons description, which constitutes a minimal field-theory description for the phase string theory.

The gauge structure of such a (2+1)-dimensional mutual-Chern-Simons theory is very unique in many aspects as compared to the gauge theories proposed before. We demonstrate that the physical symmetries, which include parity, time-reversal, and spin rotational symmetries, are precisely preserved in such an effective theory. By contrast, in usual Chern-Simons (anyon) theories[32,33,34], the parity and time-reversal symmetries are explicitly broken, including the mutual-Chern-Simons theory previously proposed[35] for describing the double-layer quantum Hall effect system.

We further show that there exist two low-temperature phases in such a theory at low doping. One is an AF state which recovers the AF long range order (AFLRO) of the Heisenberg model at half-filling and may survive at small doping concentration. The other is an SC state. Two phases are characterized by dual Meissner effects and dual flux quantization conditions, accompanied by a dual confinement, which are the direct consequences of the mutual-Chern-Simons gauge fields interacting with two matter fields when one of them experiences Bose condensation. Such a mutual duality connecting the AF and SC states or spin and charge degrees of freedom, is quite different from the usual duality descriptions proposed[36,37,38,39] for the cuprate superconductors, where the conventional boson-vortex duality is used to describe an ordered-disordered transition.

In the SC phase, for example, the Meissner effect and $hc/2e$ flux quantization are similar to the predictions by a conventional superconductivity theory, and the spinons are found to be confined such that to drop out of the physical spectrum. Only *integer* spin excitations, as composed of confined spinon pairs, are allowed in the bulk state. But as a unique prediction, a single spinon (an $S = 1/2$ moment) does appear in the center of a magnetic vortex core. It forecasts that the spin fractionalization will occur in the pseudogap phase, as the latter may be viewed as the proliferation of the vortex core state above the superconducting transition T_c[27,40].

In the AF phase, on the other hand, the spinon condensation may be viewed as a two-component "superfluidity". The dual Meissner effect means that a holon is an "alien" object in the spinon condensate, and the dual flux quantization condition corresponds to that a meron (vortex) is produced in the spinon condensate to which a holon must be confined to, just like a spinon is confined to a magnetic vortex core in the above-mentioned SC state. As a result, only

the "neutral" object of a holon-meron composite, not the holon itself, appears in the low-energy physical spectrum, which has a dipolar spin configuration at long distance, coexisting with the AFLRO in a dilute hole concentration regime.

The remainder of the paper is organized as follows. In Sec. II, we briefly introduce the effective Hamiltonian of the phase string model. In Sec. III, we first derive the Lagrangian (path-integral) formalism in the lattice version. Then we obtain the low-energy mutual-Chern-Simons gauge theory description in the continuum limit. In Sec. IV, we examine the symmetries, including parity, time-reversal, and spin rotational symmetries, of the mutual-Chern-Simons theory. In Sec. V, we study two low-temperature ordered phases based on the mutual-Chern-Simons theory and discuss how holons and spinons behave in the AF and SC phases, respectively, where dual confinement of holons/spinons is revealed. Finally, the conclusions are given in Sec. VI.

II. PHASE STRING THEORY: A MINIMAL MODEL OF DOPED ANTIFERROMAGNETS

The phase string theory has been proposed[21,23] as a low-energy effective description of the doped antiferromagnets at low doping. The "minimal" Hamiltonian of the phase string theory is composed of two terms, $H_{\text{string}} = H_h + H_s$, in which the charge degrees of freedom are characterized by the "holon" term

$$H_h = -t_h \sum_{\langle ij\rangle} \left(e^{iA^s_{ij}}\right) h^\dagger_i h_j + H.c. \tag{4}$$

where $t_h \sim t$ and the "holon" operator, $h^\dagger_i$, is bosonic; The spin degrees of freedom as described by the "spinon" term

$$H_s = -J_s \sum_{\langle ij\rangle\sigma} \left(e^{i\sigma A^h_{ij}}\right) b^\dagger_{i\sigma} b^\dagger_{j-\sigma} + H.c. \tag{5}$$

where $J_s \sim J$ and the "spinon" operator, $b^\dagger_{i\sigma}$, is also bosonic. Here the gauge fields A^s_{ij} and A^h_{ij} are decided by the topological constraints:

$$\begin{aligned} \sum_C A^s_{ij} &= \pi \sum_{l\in\Sigma_C} \left(n^b_{l\uparrow} - n^b_{l\downarrow}\right) \\ \sum_C A^h_{ij} &= \pi \sum_{l\in\Sigma_C} n^h_l \end{aligned} \tag{6}$$

where $n^b_{l\sigma}$ and n^h_l denote the "spinon" (with index σ) and "holon" number operators at site l, respectively, and the path C is an arbitrary loop made of the nearest-neighbor (nn) links with Σ_C denoting the area enclosed by C.

Basic features of this model are as follows. At half filling, the gauge field A^h_{ij} can be set to zero in (5) and H_s reduces to the Schwinger-boson mean-field Hamiltonian[16], which describes both the long-range and short-range AF correlations fairly well. Upon doping, A^h_{ij} is no longer trivial due to constraint (6), which describes that each "holon" behaves like a π-fluxoid as felt by the "spinons". Thus, A^h_{ij} will play the role of dynamic frustrations, introduced by doped holes, that acts on the spin degrees of freedom. Similarly, the "holons" are also subjected to dynamic frustrations, from the spin background, via the gauge field A^s_{ij} in (4). The spin and charge degrees of freedom are thus mutually frustrated in the phase-string model in terms of two topological gauge fields, A^h_{ij} and A^s_{ij}.

The phase string model outlined above incorporates, as a minimal model, three most essential characteristics of the doped antiferromagnets described by the $t-J$ model. They are: (i) the restricted Hilbert space of doped Mott insulators, which is characterized by the spin-charge separation formalism with holons and spinons as basic building blocks; (ii) strong short-range AF correlations as provided by the bosonic RVB description in (5), which can naturally grow into an AFLRO state as the doping concentration is reduced to zero; (iii) the mutual singular influence between the charge and spin degrees of freedom as represented by two topological gauge fields, A^h_{ij} and A^s_{ij}, which mathematically capture the phase string effect identified[23] in the $t-J$ model. Such a mutual interaction has been shown[21,27,28,29] to be responsible for some nontrivial physical properties of the model in close connection with the high-T_c materials.

In the phase string formalism, the spin operators are expressed in terms of the spinon operators in the following nontrivial form[23]

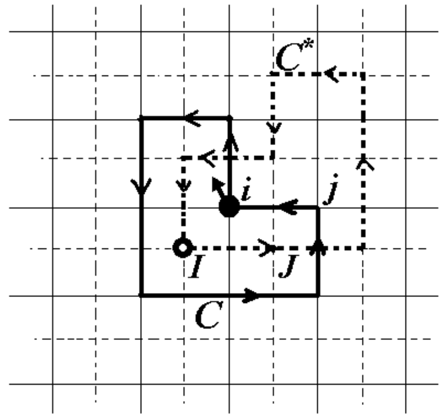

FIG. 1: A regularization of the contraints in (6) by introducing dual lattices is shown. A spinon (denoted by an arrow) and a holon (denoted by an open circle) stay in dual lattices (solid and dashed ones, respectively), with the gauge fields A^h_{ij} and A^s_{IJ} defined on the links of two dual lattices, respectively. The closed loop, C (C^*), of a spinon (holon) can be arbitary without crossing holons (spinons). See text for the detail.

$$\begin{aligned} S^z_i &= \frac{1}{2}(b^\dagger_{i\uparrow}b_{i\uparrow} - b^\dagger_{i\downarrow}b_{i\downarrow}) \\ S^+_i &= (S^-_i)^\dagger = (-1)^i b^\dagger_{i\uparrow}b_{i\downarrow}e^{i\Phi^h_i} \end{aligned} \tag{7}$$

where the phase Φ^h_i appearing in $S^\pm_i$ can be decided by the relation $\Phi^h_i - \Phi^h_j = 2A^h_{ij}$ for two nn sites, i and j, which are not occupied by the holes. Under this definition, the spin operators as well as the effective Hamiltonian are invariant under the gauge transformation $b_{i\sigma} \to b_{i\sigma}e^{i\sigma\phi_i}$, $\Phi^h_i \to \Phi^h_i + 2\phi_i$, $A^h_{ij} \to A^h_{ij} + \phi_i - \phi_j$. The holon-dependent phase factor Φ^h_i in (7) further illustrates the intrinsic mutual entanglement between spin and charge degrees of freedom in the phase string theory.

III. MUTUAL-CHERN-SIMONS GAUGE-THEORY DESCRIPTION

A. Lagrangian Formulation

The treatment of the Hamiltonian formalism of the phase string model may not be convenient beyond the mean-field approximation because the gauge fields, A^s_{ij} and A^h_{ij}, defined in (6), are themselves operators depending on the dynamics of the matter fields. In order to deal with the phase string model [(4) and (5)] more conveniently, a Lagrangian (path-integral) formalism will be introduced in this section.

First of all, let us re-express the topological constraint (6) locally. As pointed out before, the original no-double-occupancy constraint in the $t-J$ model can be realized in the phase string model by the mutual repulsion between spinons and holons via A^h_{ij} and A^s_{ij}. As a consequence, the closed path C of a holon/spinon in (6) will not cross spinons/holons and thus effectively avoid a singularity occurring when a spinon and a holon simultaneously stay at the same site (as each spinon/holon carries a π-fluxoid seen by a holon/spinon). Following this, then, it is physically reasonable to implement a regularization in the topological constraint (6) by introducing two sets of dual square lattices, respectively, for spinons and holons to stay, as illustrated in Fig. 1. In this way, a closed path C for spinon and a C^* for holon on different lattices can be arbitrary without worrying to cross the opposite species, either holons or spinons. Presumably no important low-energy physics will get lost by such a *local* regularization.

Here and below, the minuscule (majuscule) Latin letters i, j (I, J) will be used to label the dual lattice sites for spinons (holons). The Greek letters α, β, γ will be used for 2D spatial indices, 1 and 2, while μ, ν, λ for the three dimensional space-time indices, $0, 1, 2$. Then the topological constraint (6) can be re-expressed in a compact form as follows:

$$\epsilon^{\alpha\beta}\Delta_\alpha A^h_\beta(i) \;=\; \pi n^h_I$$

$$\epsilon^{\alpha\beta}\Delta_\alpha A^s_\beta(I) \;=\; \pi\sum_\sigma \sigma n^b_{i\sigma}, \tag{8}$$

in which the link fields $A^h_\alpha(i) \equiv A^h_{i+\hat{\alpha},i}$ and $A^s_\alpha(I) \equiv A^s_{I,I-\hat{\alpha}}$, with $\alpha = x,\, y$, and the difference operators, Δ_α, on the two sets of the dual lattices are defined by $\Delta_\alpha f(i) = f(i+\hat{\alpha}) - f(i)$ and $\Delta_\alpha f(I) = f(I) - f(I-\hat{\alpha})$, respectively. Note the slightly different definitions of link variables and lattice difference operators on two dual lattices, so as to keep the symmetric forms in (8).

In the path-integral formulation, the topological constraint (8) can be enforced by introducing two Lagrangian multipliers, $A^h_0(i)$ and $A^s_0(I)$, as follows

$$L_{\text{constr}} = -i\sum_I A^s_0(I)\left[n^h_I - \frac{1}{\pi}\epsilon^{\alpha\beta}\Delta_\alpha A^h_\beta(i)\right] - i\sum_i A^h_0(i)\left[\sum_\sigma \sigma n^b_{i\sigma} - \frac{1}{\pi}\epsilon^{\alpha\beta}\Delta_\alpha A^s_\beta(I)\right]. \tag{9}$$

Once the topological constraint is implemented by the Lagrangian multipliers, the gauge fields, $A^s_{I\alpha}$ and $A^h_{i\alpha}$, can be treated as *independent* gauge variables in the Lagrangian formalism. In order to get the correct form of the Lagrangian for this system, we need to first identify the canonical momenta of the gauge fields, A^h_{ij} and A^s_{IJ}.

It is helpful to consider the continuity equation for the holon density:

$$\partial_t n^h_I + \Delta^\alpha J^h_{I+\hat{\alpha},I} = 0. \tag{10}$$

Using the topological constraint in (8) and the definition of the conserved holon current, $J^h_{I+\hat{\alpha},I} = -\frac{\delta H_{\text{string}}}{\delta A^s_{I+\hat{\alpha},I}}$, one gets

$$\partial_t\left[\frac{1}{\pi}\epsilon^{\alpha\beta}\Delta_\alpha A^h_\beta(i)\right] + \Delta^\alpha\left[-\frac{\delta H_{\text{string}}}{\delta A^s_{I+\hat{\alpha},I}}\right] = 0 \tag{11}$$

such that [under a proper gauge choice of $A^h_\beta(i)$]

$$\partial_t A^h_\beta(i) = \frac{\delta H_{\text{string}}}{\delta\left(-\pi^{-1}\epsilon^{\beta\gamma}A^s_\gamma(I)\right)}. \tag{12}$$

Equation (12) is just the canonical equation of motion for $A^h_\beta(i)$, and one can thus identify the canonical momentum $\Pi^h_\beta(i) = -\frac{1}{\pi}\epsilon^{\beta\gamma}A^s_\gamma(I)$. In other words, the spatial components of the gauge fields A^h and A^s are canonically conjugate to each other. (The temporal components, A^s_0 and A^h_0, have no canonical momenta since they do not have independent dynamics in the above formulation).

Following the standard canonical quantization procedure, the *Euclidean* Lagrangian (with the Wick rotation $t \to -i\tau$) of this system can be derived straightforwardly as follows

$$\begin{aligned} L_{\text{string}} \;&=\; \sum_i \Pi^h_\alpha(i)(-i)\partial_0 A^h_\alpha(i) + \sum_i b^\dagger_{i\sigma}\partial_0 b_{i\sigma} + \sum_I h^\dagger_I\partial_0 h_I + H_{\text{string}} + L_{\text{constr}} \\ &\equiv\; L_h + L_s + L_{CS} \end{aligned}$$

where

$$\begin{aligned} L_h \;&=\; \sum_I h^\dagger_I\left[\partial_0 - iA^s_0(I)\right]h_I - t_h\sum_{\langle IJ\rangle}\left(h^\dagger_I e^{iA^s_{IJ}}h_J + h.c.\right) + \mu\left(\sum_I h^\dagger_I h_I - N\delta\right) \\ L_s \;&=\; \sum_{i\sigma} b^\dagger_{i\sigma}\left[\partial_0 - i\sigma A^h_0(i)\right]b_{i\sigma} - J_s\sum_{\langle ij\rangle\sigma}\left(b^\dagger_{i\sigma}e^{i\sigma A^h_{ij}}b^\dagger_{j-\sigma} + h.c.\right) + \lambda\left(\sum_{i\sigma} b^\dagger_{i\sigma}b_{i\sigma} - N(1-\delta)\right) \\ L_{CS} \;&=\; \frac{i}{\pi}\sum_I \epsilon^{\mu\nu\lambda}A^s_\mu(I)\partial_\nu A^h_\lambda(i) \end{aligned} \tag{13}$$

with $\partial_0 \equiv \partial_\tau$. Note that one can also use a procedure similar to (10)-(12) to define a conjugate field $\Pi^s_\alpha(I) = -\frac{1}{\pi}\epsilon^{\alpha\beta}A^h_\beta(i)$ for $A^s_\alpha(I)$ and the resulting Lagrangian remains the same as above.

Therefore, the Lagrangian formalism of the phase string model describes that the two matter fields, bosonic spinons and holons, are minimally coupled to $U(1)\times U(1)$ gauge fields, A^s_μ and A^h_μ, whose gauge structure is decided by the mutual-Chern-Simons term L_{CS} in (13). In the following, we shall further derive the long-wavelength, low-energy effective Lagrangian based on such a lattice model.

B. Low-energy effective theory

The Lagrangian (13) is written in a lattice form. It can be further simplified and reduced to a continuum version in the long-wavelength, low-energy limit. The procedure given below is quite standard and straightforward.

Let us first consider the spinon Lagrangian L_s, in which some careful treatment is needed in taking the continuum limit. We shall derive its low-energy action in the $CP(1)$ formalism[41] by integrating out the short-range *ferromagnetic* fluctuations.

First of all, we divide the square lattice into two sublattices, A and B, and redefine the spinon operator $b_{i\sigma}$ at B sublattice as $\bar{b}_{i\sigma}$. Then L_s in (13) can be rewritten as

$$\begin{aligned} L_s \;=\; & \sum_{i\in A,\sigma} b^{\dagger}_{i\sigma}(\partial_0 - i\sigma A^h_0)b_{i\sigma} + \sum_{i\in B,\sigma} \bar{b}^{\dagger}_{i\sigma}(\partial_0 - i\sigma A^h_0)\bar{b}_{i\sigma} \\ & -J_s \sum_{i\in A, j=\mathrm{nn}(i),\sigma} \left(b^{\dagger}_{i\sigma} e^{i\sigma A^h_{ij}} \bar{b}^{\dagger}_{j-\sigma} + h.c.\right) \\ & +\lambda \left(\sum_{i\in A,\sigma} b^{\dagger}_{i\sigma} b_{i\sigma} + \sum_{i\in B,\sigma} \bar{b}^{\dagger}_{i\sigma}\bar{b}_{i\sigma} - N\left(1-\delta\right)\right). \end{aligned} \tag{14}$$

As usual, we introduce the following continuum fields

$$\begin{aligned} b_{i\sigma} &= z_\sigma(\mathbf{r}_i) + \pi_\sigma(\mathbf{r}_i) \\ \bar{b}^{\dagger}_{i+\hat{\eta},-\sigma} &= z_\sigma(\mathbf{r}_i + \hat{\eta}a) - \pi_\sigma(\mathbf{r}_i + \hat{\eta}a) \end{aligned} \tag{15}$$

in which $i \in A$, $\hat{\eta} = \hat{x}, \hat{y}$, and a is the lattice constant. Then, by expressing the Lagrangian (14) in terms of z_σ and π_σ and taking the continuum limit $a \to 0$ with $A^h_\alpha(i) \to aA^h_\alpha(\mathbf{r})$, we obtain $L_s = \int d^2\mathbf{r}\mathcal{L}_s$, in which

$$\begin{aligned} \mathcal{L}_s \;=\; & \sum_{\sigma} \left[J_s \left| \left(\partial_\alpha - i\sigma A^h_\alpha\right) z_\sigma \right|^2 + a^{-2}\left(\lambda - 4J_s\right)\left|z_\sigma\right|^2 \right] - \lambda a^{-2}\left(1-\delta\right) \\ & + \sum_{\sigma} \left[-J_s \left| \left(\partial_\alpha - i\sigma A^h_\alpha\right) \pi_\sigma \right|^2 + a^{-2}\left(\lambda + 4J_s\right)\left|\pi_\sigma\right|^2 \right] \\ & + a^{-2} \sum_{\sigma} \left[\pi^*_\sigma \left(\partial_0 - i\sigma A^h_0\right) z_\sigma - \pi_\sigma \left(\partial_0 + i\sigma A^h_0\right) z^*_\sigma \right] \end{aligned} \tag{16}$$

By further integrating out the high-energy field π^σ, we arrive at

$$\mathcal{L}_s = \sum_{\sigma} \left(\frac{a^{-2}}{\lambda + 4J_s} \left| \left(\partial_0 - i\sigma A^h_0\right) z_\sigma \right|^2 + J_s \left| \left(\partial_\alpha - i\sigma A^h_\alpha\right) z_\sigma \right|^2 + a^{-2}\left(\lambda - 4J_s\right)\left|z_\sigma\right|^2 \right) - \lambda a^{-2}\left(1-\delta\right) \tag{17}$$

Define the spin-wave velocity $c_s = \sqrt{J_s(\lambda + 4J_s)}a$ and redefine the temporal components: $\tau \to c_s x_0$, $A^h_0 \to c_s A^h_0$, the low-energy effective action for the spinons can be finally written as

$$S_s = \int d^2\mathbf{r} \int_0^{c_s\beta} dx_0 \frac{1}{2g}\left[|(\partial_\mu - i\sigma A^h_\mu) z_\sigma|^2 + m_s^2 |z_\sigma|^2 \right], \tag{18}$$

Here the summations over $\mu = 0, 1, 2$ and $\sigma = \uparrow, \downarrow$ are omitted for simplicity and the constant term $\lambda a^{-2}\left(1-\delta\right)$ is also dropped. The coupling constant $g = c_s/2J_s\left(1-\delta\right)$, and the mass $m_s = c_s^{-1}\sqrt{\lambda^2 - 16J_s^2}$, in which λ is decided by the spinon number constraint $\int d^2x \sum_\sigma \left|z_\sigma\right|^2 = Na^2$. Note that here the z_σ field has been rescaled in the last step such that $\sum_\sigma \left|z_\sigma\right|^2$ keeps to be 1 per site on average even at finite doping. Therefore, in its final form, the long-wavelength theory for spinons consists of a massive, spin-1/2, and relativistic bosonic z_σ (spinon) coupled to a $U(1)$ gauge field A^h_μ.

The continuum versions of L_h and L_{CS} can be more straightforwardly obtained by directly taking the continuum limit $a \to 0$, with $A^s_\alpha(I) \to aA^s_\alpha(\mathbf{r})$, $A^s_0(I) \to c_s A^s_0(\mathbf{r})$, and $h_I \to ah(\mathbf{r})$. The final form of the partition function can be written in the compact form

$$Z = \int DhDz_\uparrow Dz_\downarrow DA^s DA^h \exp\left(-\int_0^{c_s\beta} dx_0 \int d^2\mathbf{r}\mathcal{L}_{\mathrm{eff}} \right)$$

in which

$$\mathcal{L}_{\rm eff} = \mathcal{L}_h + \mathcal{L}_s + \mathcal{L}_{CS} \tag{19}$$

with

$$\begin{aligned} \mathcal{L}_h &= h^\dagger \left[\partial_0 - i(A_0^s + eA_0^e)\right] h + h^\dagger \frac{(-i\partial_\alpha - A_\alpha^s - eA_\alpha^e)^2}{2m_h} h \\ \mathcal{L}_s &= \frac{1}{2g}\left[|(\partial_\mu - i\sigma A_\mu^h)z_\sigma|^2 + m_s^2 |z_\sigma|^2\right] \\ \mathcal{L}_{CS} &= \frac{i}{\pi}\epsilon^{\mu\nu\lambda} A_\mu^s \partial_\nu A_\lambda^h \end{aligned} \tag{20}$$

where in the holon Lagrangian density $\mathcal{L}_h$, $m_h \simeq (2t_h a^2)^{-1}$, A_μ^e is the vector potential of the external electromagnetic field, and $-e$ is the electron electric charge. Note that the chemical potential μ in $\mathcal{L}_h$ has been absorbed into iA_0^s for simplicity.

The Lagrangians in (20) constitute our final low-energy effective theory. They describe two matter fields, holons and spinons, minimally coupled to a pair of U(1)×U(1) gauge fields, A_μ^s and A_μ^h. The latter do not have their own kinetic energies, but are mutually "entangled" by the mutual-Chern-Simons term $\mathcal{L}_{CS}$. Such a mutual-Chern-Simons term has been previously proposed[35] for describing the double-layer quantum Hall effect system. But here due to the fact that A_μ^h couples to up/down spins with opposite "charges" in $\mathcal{L}_s$, the parity and time-reversal symmetries are explicitly retained (see below). The external electromagnetic field, A_μ^e, only directly couples to the holon field, indicating that the latter is the primary charge carrier (consistent with the definition of the holon). This is in contrast to the usual U(1) gauge theory based on the slave-boson approach, in which both holon and spinon share the external electromagnetic field as if each of them carriers a fractional part of the charge e (as the result that both of them see the same internal U(1) gauge field).

In the following section, we shall carefully examine the symmetries of this effective Lagrangian with a particular attention to the parity, time-reversal, and spin SU(2) rotational symmetries.

IV. SYMMETRIES

The symmetries of the present mutual-Chern-Simons Lagrangian will be studied in this section. The following discussions will be based on the low-energy effective Lagrangian (20), although all of them can be easily generalized to the lattice formalism in (13).

First of all, we note that the $\mathrm{U(1)_{charge}} \times \mathrm{U(1)_{S_z}}$ gauge invariance of $S_{\rm eff}$ is obvious according to (20). Consequently, the global $\mathrm{U(1)_{charge}}$ invariance of the holons ensures the conservation of the electromagnetic charge in this system. Also straightforward is the translational invariance in (2+1)-dimensions. In the following, we shall mainly focus on the parity, time-reversal, and spin rotational symmetries, and show that they are explicitly retained in the present mutual-Chern-Simons gauge theory, in contrast to ordinary Chern-Simons theories in which the parity and time-reversal symmetries are usually broken.

A. Parity

In (2+1)-dimensions, the parity transformation is defined as a reflection with regard to a spatial axis, *e.g.*,

$$x \to -x,\ y \to y,\ \tau \to \tau \tag{21}$$

It is straightforward to verify that the effective Lagrangians, $\mathcal{L}_h$, $\mathcal{L}_s$, and $\mathcal{L}_{CS}$, remain invariant, respectively, under the parity transformation (21), if the matter fields and gauge fields transform under (21) as follows

$$\begin{aligned} z_\sigma &\to z_{-\sigma},\quad h \to h \\ A_0^h &\to -A_0^h,\ A_x^h \to A_x^h,\ A_y^h \to -A_y^h \\ A_0^s &\to A_0^s,\ A_x^s \to -A_x^s,\ A_y^s \to A_y^s \end{aligned} \tag{22}$$

The parity transformations of the fields in (22) can be determined as follows. For example, according to the property of angular momenta, a spin should transform as an axial vector, namely, $S_x \to S_x$, $S_y \to -S_y, S_z \to -S_z$ under the

parity transformation (21). Thus the transformation of the $CP(1)$ field z_σ should be $z_\sigma \to z_{-\sigma}$. On the other hand, the gauge field A^h transforms as an axial vector and A^s as a polar vector in (22). Indeed, in order to keep the invariance of $\mathcal{L}_h$ and $\mathcal{L}_s$, the parity of A^s_μ and A^h_μ should be identical to the charge current $j^h_\mu = -\delta\mathcal{L}_h/\delta A^s_\mu$ and spin current $j^s_\mu = -\delta\mathcal{L}_s/\delta A^h_\mu$, respectively. Furthermore, the parity transformations of A^s_μ and A^h_μ are also consistent with the classical equations of motion for the Chern-Simons fields obtained based on (20):

$$j^s_\mu = \frac{i}{\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A^s_\lambda,\ j^h_\mu = \frac{i}{\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A^h_\lambda. \tag{23}$$

The parity invariance of the mutual Chern-Simons term is also related to the fact that the gauge field A^h transforms as an axial vector and A^s as a polar vector, in contrast to an ordinary U(1) Chern-Simons theory.

B. Time-reversal

Under the time-reversal transformation,

$$\tau \to -\tau,\ r_\alpha \to r_\alpha, \tag{24}$$

the z_σ and h field will transform as usual spinor and scalar fields, respectively. Using the same procedure as given above in the parity transformation, we can determine

$$\begin{aligned} z_\sigma &\to \sigma z^*_{-\sigma},\ h \to h^* \\ A^h_0 &\to -A^h_0,\ A^h_\alpha \to A^h_\alpha \\ A^s_0 &\to A^s_0,\ A^s_\alpha \to -A^s_\alpha \end{aligned} \tag{25}$$

under the time-reversal transformation (24). It can be then easily checked that the Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L}_h + \mathcal{L}_s + \mathcal{L}_{CS}$ is also invariant under the time-reversal transformation.

The parity and time-reversal invariances of the mutual Chern-Simons Lagrangian (20) are in sharp contrast to the violations of both, separately, in an ordinary U(1) Chern-Simons theory. As noted above, A^h as an axial vector and A^s as a polar vector in the mutual U(1) × U(1) Chern-Simons theory are the key for the restoration of the symmetries. Note that the charge conjugate symmetry is meaningless here since the holon Lagrangian $\mathcal{L}_h$ is non-relativistic and anti-holons are not well defined.

C. Spin SU(2) rotation

The demonstration of the global spin SU(2) symmetry in the present formulation is less straightforward than the other symmetries discussed above. The underlying reason is that the spin operators are expressed in an unconventional way in terms of the $b_i \equiv (b_{i\uparrow}, b_{i\downarrow})^T$ doublet according to (7).

Let us consider a global SU(2) spin rotation defined by $U = \exp\left(i\boldsymbol{\theta}\cdot\mathbf{S}\right)$. In terms of (7), one finds $U^{-1}b_iU = (\sigma_3)^i e^{-i\sigma_3\Phi^h_i/2}e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}e^{i\sigma_3\Phi^h_i/2}(\sigma_3)^i b_i$. Correspondingly, according to the definition of the $CP(1)$ fields in (15), the doublet $z = (z_\uparrow, z_\downarrow)^T$ under the $SU(2)$ rotation U is given by

$$U^{-1}z(\mathbf{r},\tau)U = e^{i\sigma_3\Phi^h(\mathbf{r},\tau)/2}e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2}e^{-i\sigma_3\Phi^h(\mathbf{r},\tau)/2}z(\mathbf{r},\tau) \tag{26}$$

in which

$$\partial_\mu\Phi^h(\mathbf{r},\tau) = 2A^h_\mu(\mathbf{r},\tau). \tag{27}$$

(Note that in the Hamiltonian formalism, the single-valueness of Φ^h_i in the spin operators (7) is ensured by the topological constraint on A^h_{ij} according to (6). In the path-integral formalism, Φ^h is determined by (27), and we show in Appendix A that to have a finite contribution to the partition function, Φ^h must still satisfy the single-valueness constraint: $\Delta\Phi^h|_C \equiv \oint_C \partial_\mu\Phi^h dx_\mu = \oint_C 2A^h_\mu dx_\mu = 2n\pi$, with $n \in \mathbb{Z}$ for an arbitrary loop C.)

The spinon Lagrangian $\mathcal{L}_s$ can be rewritten as

$$\mathcal{L}_s = \frac{1}{2g}\left\{(D_\mu z)^\dagger D_\mu z + m_s^2 z^\dagger z\right\} \tag{28}$$

in which $D_\mu z \equiv \left(\partial_\mu - i\sigma_3 A_\mu^h\right) z = \partial_\mu \left(e^{-i\sigma_3\Phi^h/2} z\right)$. Under the transformation (26), $D_\mu z$ transforms as

$$U^{-1}\left(D_\mu z\right) U = e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2} D_\mu z.$$

Namely, $D_\mu z$ transforms as the basic representation of the SU(2) group, and the SU(2) invariance of the Lagrangian (28) is proved. Independent of $z(\mathbf{r}, \tau)$, Lagrangians $\mathcal{L}_h$ and $\mathcal{L}_{CS}$ are obviously invariant. Therefore, the global spin $SU(2)$ symmetry is indeed preserved in the present mutual-Chern-Simons theory.

V. TWO ORDERED PHASES AT LOW TEMPERATURES

A. AF phase at low doping

1. *Half-filling*

Let us first consider Lagrangian (19) at half filling. Without the presence of holons, one can find $A_\mu^h = 0$ and $\mathcal{L}_{\text{eff}}$ reduces to a CP(1) model

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_s = \frac{1}{2g}\left\{|\partial_\mu z|^2 + m_s^2 |z|^2\right\}. \tag{29}$$

The saddle-point solution of (29) can be obtained by a standard procedure after integrating out the CP(1) z-field and then minimizing the resulting action with regard to m_s^2 (here the constant term $-m_s^2/2g$ previously dropped in $\mathcal{L}_s$ has to be included) as follows[42,43,44]

$$gT\sum_{\omega_n}\int \frac{d^2\mathbf{k}}{4\pi^2}\frac{1}{\mathbf{k}^2 + \omega_n^2 + m_s^2} = 1 \tag{30}$$

where $\omega_n = 2\pi n T$, $n =$ integers. With a proper regularization[43] in (30), the mass gap m_s can be determined at small T as

$$m_s \approx T\exp(-\frac{2\pi}{T}\frac{1}{\tilde{g}}). \tag{31}$$

in the so-called renormalized classical region, where $\frac{1}{\tilde{g}} \equiv \frac{1}{g} - \frac{1}{g_c} > 0$ (here $g_c = \frac{4\pi}{\Lambda}$ with Λ denoting a cutoff parameter in the regularization).

At $T = 0$, the mass gap $m_s = 0$, and a Bose condensation takes place in the ground state with $\langle z \rangle \neq 0$, corresponding to an AFLRO lying in the x-y plane: $\left\langle S_i^+ \right\rangle = (-1)^i < z_\uparrow >< z_\downarrow >$, which can be easily destroyed by thermal fluctuations at any finite temperatures as indicated by $m_s > 0$ according to (31).

The energy scale of the mass gap m_s is always much smaller than the temperature, *i.e.,* $m_s \ll T$, at $T \ll \frac{1}{\tilde{g}}$. Thus, $\omega_n = 2\pi n T$ $(n \geq 1)$ is usually much larger than the mass gap, which means that the quantum fluctuations will become negligible in a sufficiently long wavelength and low energy regime, where one may only consider the purely static (semiclassical) fluctuations. In the region of $m_s < k < c_s\beta$, the effective Lagrangian of the CP(1) field will lose the Lorentz invariance and becomes

$$\mathcal{L}_s \approx \frac{1}{2\tilde{g}}|\boldsymbol{\nabla} z|^2. \tag{32}$$

Such an effective Lagrangian can be also obtained in the renormalized classical region by using the O(3) nonlinear σ model[42].

2. *Low doping*

In a sufficiently small concentration of holes, if the AFLRO or the Bose condensation of the CP(1) spinor fields persists, then the renormalized classical Lagrangian (32) remains applicable, which should be simply modified to couple to the gauge field $\mathbf{A}^h$ according to (20) as follows:

$$\mathcal{L}_s = \frac{1}{2\tilde{g}}|(\boldsymbol{\nabla} - i\sigma_3\mathbf{A}^h)z|^2. \tag{33}$$

On the other hand, holons are coupled to A_μ^s in $\mathcal{L}_h$, and two gauge fields are then entangled by the mutual-Chern-Simons term $\mathcal{L}_{CS}$ [see (20)], which can be rewritten, up to a boundary term, as

$$\mathcal{L}_{CS} = -\frac{i}{\pi}\mathbf{A}^h \cdot (\mathbf{E}^s \times \hat{\mathbf{z}}) + \frac{i}{\pi}A_0^h B^s \tag{34}$$

where we introduce $\mathbf{E}^s \equiv \partial_0 \mathbf{A}^s - \nabla A_0^s$ as the "electric field" strength for A_μ^s and $B^s = \nabla \times \mathbf{A}^s \cdot \hat{\mathbf{z}}$ as its "magnetic field" strength. By integrating out A_μ^h, then, the spin dynamics will become entangled with the holon dynamics as shown below.

First of all, the integration over A_0^h will simply lead to $B^s = 0$. In the following, one may then choose a proper gauge: $\mathbf{A}^s = 0$ and $\mathbf{E}^s = -\nabla A_0^s$. Next, by using

$$|(\boldsymbol{\nabla} - i\sigma_3 \mathbf{A}^h)z|^2 = |\boldsymbol{\nabla}\tilde{z}|^2 + 2\mathbf{A}^h \cdot \mathbf{v}^s + \left(\mathbf{A}^h\right)^2 |\tilde{z}|^2$$

with $\tilde{z} \equiv \left(z_\uparrow, z_\downarrow^*\right)^{\mathrm{T}}$ and $\mathbf{v}^s \equiv \frac{i}{2}\left(\tilde{z}^\dagger \nabla \tilde{z} - \nabla \tilde{z}^\dagger \tilde{z}\right)$, one has

$$\mathcal{L}_s + \mathcal{L}_{CS} = \frac{1}{2\tilde{g}}|\boldsymbol{\nabla}\tilde{z}|^2 + \mathbf{A}^h \cdot \left(\frac{1}{\tilde{g}}\mathbf{v}^s - \frac{i}{\pi}\mathbf{E}^s \times \hat{\mathbf{z}}\right) + \frac{1}{2\tilde{g}}\left(\mathbf{A}^h\right)^2,$$

under the constraint $|\tilde{z}|^2 = 1$, which, after integrating out $\mathbf{A}^h$, arrives at

$$\begin{aligned} &\frac{1}{2\tilde{g}}|\boldsymbol{\nabla}\tilde{z}|^2 - \frac{\tilde{g}}{2}\left(\frac{1}{\tilde{g}}\mathbf{v}^s - \frac{i}{\pi}\mathbf{E}^s \times \hat{\mathbf{z}}\right)^2 \\ =&\frac{1}{2\tilde{g}}\left(|\boldsymbol{\nabla}\tilde{z}|^2 - |\mathbf{v}^s|^2\right) + \frac{\tilde{g}}{2\pi^2}(\mathbf{E}^s)^2 + \frac{i}{\pi}(\mathbf{E}^s \times \hat{\mathbf{z}}) \cdot \mathbf{v}^s. \end{aligned} \tag{35}$$

Finally, by introducing a unit vector $\tilde{\mathbf{n}}$ defined by

$$\tilde{\mathbf{n}} = \tilde{z}^\dagger \boldsymbol{\sigma} \tilde{z}$$

and by using

$$\frac{1}{4}|\boldsymbol{\nabla}\tilde{\mathbf{n}}|^2 = |\boldsymbol{\nabla}\tilde{z}|^2 - |\mathbf{v}^s|^2$$

the low-energy effective Lagrangian reduces to

$$\mathcal{L}_{\text{eff}} = \frac{1}{8\tilde{g}}(\boldsymbol{\nabla}\tilde{\mathbf{n}})^2 + \frac{\tilde{g}}{2\pi^2}(\mathbf{E}^s)^2 + iA_0^s \mathcal{K}_0^s + \mathcal{L}_h \tag{36}$$

where

$$\begin{aligned} \mathcal{K}_0^s &\equiv \frac{1}{\pi}\epsilon_{0\nu\lambda}\partial_\nu v_\lambda \\ &= \frac{1}{4\pi}\epsilon_{0\nu\lambda}\tilde{\mathbf{n}} \cdot \partial^\nu \tilde{\mathbf{n}} \times \partial^\lambda \tilde{\mathbf{n}}. \end{aligned}$$

This low-energy Lagrangian describes how the bosonic holons, via $\mathcal{L}_h$, and spin twists, with topological charge density $\mathcal{K}_0^s$, are coupled to a Maxwell gauge field A^s *with the "photon velocity"* $c = \infty$, that is, in the absence of $|\mathbf{B}^s|^2$. The only effect of such a non-relativistic gauge field is then to induce a 2D Coulomb interaction between two types of charged particles, including holons and spin twists characterized by $\mathcal{K}_0^s$. Noting $\mathbf{E}^s = -\nabla A_0^s$ and integrating out A_0^s in (36), a potential term will emerge in the effective action as

$$V = q_h^2 \int d^2\mathbf{r} d^2\mathbf{r}' \ln|\mathbf{r} - \mathbf{r}'| \left(\rho_h + \mathcal{K}_0^s\right)(\mathbf{r}) \left(\rho_h + \mathcal{K}_0^s\right)(\mathbf{r}') \tag{37}$$

in which $\rho_h = h^\dagger h$ and $q_h^2 = \pi^3/\tilde{g}^2$, accompanied by a *charge neutral* constraint enforced in the thermodynamic limit on the low energy states of such a 2D Coulomb gas system, namely

$$\int d^2\mathbf{r}\left[\rho_h(\mathbf{r}) + \mathcal{K}_0^s(\mathbf{r})\right] = 0. \tag{38}$$

Thus, a holon has to be "confined" to a spin twist, satisfying $1 + \int d^2\mathbf{r}\frac{1}{2\pi}\tilde{\mathbf{n}}\cdot\partial_x\tilde{\mathbf{n}}\times\partial_y\tilde{\mathbf{n}} = 0$, which leads to the quantization condition of the winding number of the unit vector $\{\tilde{\mathbf{n}}(\mathbf{r})\}$ in spin space as follows

$$\begin{aligned} \mathcal{Q}^s &\equiv \int d^2\mathbf{r}\frac{1}{4\pi}\tilde{\mathbf{n}}\cdot\partial_x\tilde{\mathbf{n}}\times\partial_y\tilde{\mathbf{n}} \\ &= -\frac{1}{2}. \end{aligned} \tag{39}$$

Namely, each holon will be bound to a "meron", which is a spin twist of the unit vector $\tilde{\mathbf{n}}$ whose winding number is half of that for a Skyrmion.

According to the condition (38), one expects to find equal number of holons and (anti)merons at low temperatures, which are paired by the logarithmic-attractive interaction in (37). An unpaired holon or (anti)meron will cost a logarithmically divergent energy and thus is forbidden to appear. In other words, in the AF phase, a bare holon can not exist alone, but has to be always confined to a spin topological configuration (meron). Such an effect in the spin ordered phase is called the "*holon confinement*". Note that a holon itself will also carry a spin vortex according to (7), the composite object formed by the holon-meron pair actually corresponds to a spin dipolar configuration in the *real* spin space, as previous identified in the phase string model[28,29]. Since the (anti)meron is a semiclassical object without a coherent quantum dynamics, the dipole as a bound pair of a holon and a (anti)meron normally cannot move coherently either. That is, the holon will be *self-trapped* near the core of the meron in space and the translation symmetry is spontaneously broken.

With the increase of doping, *i.e.,* the number of holon-antimeron dipoles, one expects to see a screening effect on the confining potential V. It has been previously found that eventually a confinement-deconfinement transition can take place beyond some critical doping concentration, where the screened 2D Coulomb interaction becomes short-ranged[28,29]. Once the bosonic holons are free, they will experience a Bose condensation and the resulting phase is an SC state as to be discussed in the following section. In the SC phase, there exists a duality correspondence of the quantization condition (39), which will ensures the flux quantization condition there. Correspondingly, (39) may be called a *dual flux-quantization condition.*

Finally we remark that the hole self-trapping at low doping, discussed in the present work, is in contrast to a conventional picture for single hole moving in the AF background based on the numerical studies of the $t-J$ model[45]. In the latter case, the doped hole is found to have finite spectral weight and a coherent dispersion with the bandwidth comparable to J. The discrepancy may arise from the small sample sizes in exact diagonalization calculations: The phase string effect, which leads to the mutual Chern-Simons gauge fields, starts to play the role of self-localization only when the sample sizes become larger than the localization length scales[46]. Further investigations from both analytic and numerical approaches to clarify this issue are needed. Possible experimental implications of self-trapping for lightly doped cuprate have been previously discussed in the phase string model[29].

B. Meissner effect and spinon confinement in SC phase

Now let us consider the other ordered phase with the Bose condensation of holons, $\langle h\rangle \neq 0$, whose ground state is a superconducting one[27] with the Meissner effect and charge $2e$ minimal flux quantization as shown below.

With $\langle h\rangle \neq 0$, $\mathcal{L}_h$ in (20) reduces to

$$\mathcal{L}_h = i\rho_h(\partial_0\phi_h - A_0^s) + \frac{\rho_h}{2m_h}\left(\boldsymbol{\nabla}\phi_h - \mathbf{A}^s - \mathbf{A}^e\right)^2 \tag{40}$$

by writing $h(\mathbf{r}) = \sqrt{\rho_h}e^{i\phi_h(\mathbf{r})}$. The Chern-Simons term (20) can be rewritten as

$$\mathcal{L}_{\rm CS} = -\frac{i}{\pi}\mathbf{A}^s\cdot(\mathbf{E}^h\times\hat{\mathbf{z}}) + \frac{i}{\pi}A_0^s\mathbf{B}^h\cdot\hat{\mathbf{z}} \tag{41}$$

by introducing the "electric" field $\mathbf{E}^h = \partial_0\mathbf{A}^h - \nabla A_0^h$ and "magnetic" field $\mathbf{B}^h = \nabla\times\mathbf{A}^h$ for the vector potential $\mathbf{A}^h$.

Firstly, the "magnetic" field $B^h = \mathbf{B}^h\cdot\hat{\mathbf{z}}$ can be determined after integrating out A_0^s in the partition function and one obtains the condition

$$B^h = \mathbf{B}^h\cdot\hat{\mathbf{z}} = \pi\rho_h \tag{42}$$

which is uniform and fixes the spatial component $\mathbf{A}^h$, such that $\mathbf{E}^h = -\nabla A_0^h$. Secondly, after integrating out $\mathbf{A}^s$, the resulting effective Lagrangian takes the following form

$$\mathcal{L}_{\rm eff} = \mathcal{L}_s + \left(\frac{m_h}{2\pi^2\rho_h}\right)\left|\mathbf{E}^h\right|^2 - iA_0^h\mathcal{Q}^h \tag{43}$$

in which

$$\mathcal{Q}^h \equiv \frac{1}{\pi}\epsilon^{0\nu\lambda}\partial_\nu\left(\partial_\lambda\phi_h - A^e_\lambda\right).$$

Finally, we integrate out A_0^h in (43). For our purpose, instead of using the continuous version (20) of $\mathcal{L}_s$, we shall use a simpler but more precise form of the term involving A_0^h based on the original L_s defined in (13), which reads

$$\mathcal{L}_s = -iA_0^h\rho_s(\mathbf{r}) + \mathcal{L}_s\left(A_0^h = 0\right)$$

in which $\rho_s(\mathbf{r}) = \rho_\uparrow(\mathbf{r}) - \rho_\downarrow(\mathbf{r})$ with $\rho_\uparrow(\mathbf{r})[\rho_\downarrow(\mathbf{r})]$ denotes the density of up (down) spinons. Then, after integrating out A_0^h, one obtains the following effective action in (2+1)-dimensional Euclidean space

$$S_{\text{eff}} = \int d^3x_\mu\left[\mathcal{L}_s\left(A_0^h = 0\right)\right] + \int dx_0 V_{\text{SC}}$$

where

$$V_{\text{SC}} = q_s^2\int d^2\mathbf{r}d^2\mathbf{r}'\ln|\mathbf{r}-\mathbf{r}'|\left(\rho_s + \mathcal{Q}^h\right)(\mathbf{r})\left(\rho_s + \mathcal{Q}^h\right)(\mathbf{r}') \tag{44}$$

with $q_s^2 = \frac{\pi\rho_h}{4m_h}$. Similar to the case in the AF phase, in the thermodynamic limit, there is a charge neutral condition enforced as follows:

$$\begin{aligned} 0 &= \int d^2\mathbf{r}\left[\rho_s(\mathbf{r}) + \mathcal{Q}^h(\mathbf{r})\right] \\ &= N_\uparrow - N_\downarrow + \left(2N_{\text{vor}} - \frac{\Phi^e}{\pi}\right), \end{aligned} \tag{45}$$

in which $N_\uparrow$ ($N_\downarrow$) is the total number of spin-up (spin-down) spins, $N_{\text{vor}} = \frac{1}{2\pi}\int d^2\mathbf{r}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta\phi_h$ denotes the total number of 2π vortices in the holon field, and Φ^e is the total external magnetic flux $\Phi^e = \int d^2\mathbf{r}\epsilon^{0\nu\lambda}\partial_\nu A^e_\lambda$.

Since $N_\uparrow$, $N_\downarrow$, and N_{vor} are all quantized to be integers, we find the minimal *flux quantization condition*

$$|\Phi^e_{\min}| = \frac{\Phi_0}{2} \tag{46}$$

where $\Phi_0 = 2\pi$ ($= hc/e$ in full units) is the flux quantum for a charge e system.

Therefore, the external magnetic flux is not allowed to present in the bulk (i.e., the Meissner effect) unless it is quantized in multiples of half flux quanta given in (46). In particular, for a magnetic flux quantized at the minimal half flux quantum $\Phi^e_{\min}$, there must be a spinon trapped near the vortex core according to (45) by noting that the "charge" $2N_{\text{vor}}$ of the vortices produced by holon field is always in units of 2π (i.e., Φ_0).

However, free spinons are not allowed in the bulk in the absence of the external magnetic flux. Indeed, for $\Phi^e = 0$, the "charge neutral" condition (45) reduces to $N_\uparrow - N_\downarrow + 2N_{\text{vor}} = 0$. As the result, a single spinon excitation, with $S^z = (N_\uparrow - N_\downarrow)/2 = \pm 1/2$, will violate the "charge neutral" condition, which in fact will cost a logarithmically divergent energy as each spinon behaves like a half vortex. Hence, in the superconducting state, the spinons-vortices must be always paired up (confined) in the bulk by the logarithmic force given in (44). To be noted, not only the spinons with different spin indices (up/down) can pair up, those with the same spin indices (up/up and down/down) can also pair up to satisfy the charge neutral condition by involving a holon phase vortex with $N_{\text{vor}} \neq 0$, *e.g.,* $N_{\text{vor}} = 1, N_\uparrow = 2, N_\downarrow = 0$ or $N_{\text{vor}} = -1, N_\uparrow = 0, N_\downarrow = 2$. Since N_{vor} does not appear in the rest of the action, these two excitations of $S^z = \pm 1$ are energy degenerate with the state $S^z = 0$, $N_{\text{vor}} = 0$, to form $\mathbf{S} = 1$ triplet spin excitations, consistent with the spin rotation symmetry generally demonstrated before. For the same reason, the single spinon bound to a magnetic vortex quantized at $\frac{\Phi_0}{2}$ should have a free moment with $S^z = \pm 1/2$, because of the freedom introduced by N_{vor}.

Thus, the superconducting phase in the present mutual-Chern-Simons theory is characterized by the holon condensation and spinon (logarithmic) confinement. We have seen that the fractionalization of spins (a single spinon) does not directly appear in the bulk low-lying excitation spectrum, but does show up in a magnetic vortex core[27]. The deconfinement of spinon-vortex pairs will eventually occur at the superconducting transition temperature T_c[27,40]. Finally, we point out that the symmetry of the superconducting order parameter, which is expressed in terms of the electron operator (3), is d-wave like as discussed previously in Ref.[47].

C. Mutual duality of two phases

The doping effect and the interplay between charge and spin degrees of freedom are characterized by a mutual-Chern-Simons gauge structure in this model, as discussed in previous sections. The mutually dual characteristics of these two phases are summarized by the following table.

	AF	SC
Bose condensation	$<\mathbf{z}>\neq 0$	$<h>\neq 0$
Coulomb gauge field	A_0^s	A_0^h
"charged" particle of Coulomb gauge field	holon	spinon
external source of Coulomb gauge field	meron	magnetic flux
"charge neutral" object	holon-meron pair	a. spinon pair b. magnetic flux + a spinon
dual flux quantization	$\lvert\mathcal{Q}^s\rvert = \frac{1}{2}$	$\lvert\Phi^e_{\text{min}}\rvert = \frac{\Phi_0}{2} = \frac{hc}{2e}$
dual Meissner effect	holon confinement	a. spinon confinement b. spinon bound to magnetic flux

We have shown that, at low doping, the spinon condensation leads to a spin AF order and forces a "confinement" on the holon part, making holons self-localized to ensure the AFLRO. On the other hand, at a higher doping, the condensation of bosonic holons forces a "confinement" on the spinon part, resulting an SC phase coherence.

There are several distinctions between the two ordered phases. In the AF phase, the spinons condensate is a kind of two-component "superfluid". Consequently the global symmetry is broken from SU(2) to U(1). In contrast, the ground state in the SC phase is a condensation of a scalar field - holons. As a result the global U(1) symmetry of the charge part is broken.

Besides such a fundamental distinction, two phases share some common features originated from the duality in the mutual-Chern-Simons gauge structure. In both the AF and SC phases, there exist induced Maxwell terms that have only "electric field strengths" without the Lorentz invariance. There are "charges" coupling to these Coulomb gauge fields, including quantum particles (holons and spinons) and external sources without quantum dynamics. The "charge neutral" condition and the 2D Coulomb interaction among the "charged" objects lead to dual Meissner effects: In the SC case, an external magnetic flux as an external source must be quantized, and in order to realize a minimal quantum $hc/2e$, a single spinon must be bound to such a magnetic flux to form a "charge neutral" object; In the AFLRO state, as an external source, a (anti)meron is allowed as a topological excitation from the spinon condensate with a quantized winding number $|\mathcal{Q}^s| = \frac{1}{2}$ and a holon cannot live alone and must be bound to such an external source to form a "charge neutral" object with a spin dipolar configuration.

Finally we emphasize that the mutual-Chern-Simons theory in this work involves a mutual duality between the charge and spin degrees of freedom rather than a usual duality. A usual dual description has been also widely used[36,37,38,39] in studying the doped Mott insulators, which deals with an ordered phase and the transition to a disordered phase in terms of the corresponding topological defects on the dual lattice. In a conventional dual-theory description, normally the AF and SC phases are not directly related. By contrast, in the mutual duality discussed in the present theory, the vortices of one species (holon/spinon) under condensation are themselves quantum objects of another species (spinon/holon), and *two* ordered phases, i.e., AF and SC states, can be naturally unified together[28]. A similar duality at low doping has been also investigated[48] by starting from the slave-fermion approach[15]. Based on the same phase string decomposition (3) but with a slightly different mean-field decoupling (see Ref.[25]), a mutual duality between the AF and SC states has been also discussed recently in Ref.[24].

VI. CONCLUSION

In this paper, we studied a new class of nontrivial (2+1)-dimensional gauge field structure - the mutual-Chern-Simons theory. The Lagrangian of such a mutual-Chern-Simons theory is derived as an effective low-energy description of the phase-string model for doped Mott insulators. This effective Lagrangian retains the full symmetries of parity, time-reversal, and global SU(2) spin rotation, in contrast to the conventional Chern-Simons theories where first two symmetries are usually broken.

The mutual-Chern-Simons theory as a minimal model for doped Mott insulators has a unique mutual duality structure. Two ordered phases found in this theory, the AF and SC states, are connected by dual Meissner/flux quantization effects and dual confinement/deconfinement. Namely, holons become vortices in the spinon condensed AF phase and spinons become vortices in the holon condensed SC state. The former leads to the holon confinement (a holon bound to a spin meron twist to form a "neutral" dipolar structure) and the latter leads to the spinon confinement and flux quantization (a spinon bound to a magnetic flux quantized at $hc/2e$ to form a "neutral" object in the holon condensate).

Such a mutual duality structure between the charge and spin degrees of freedom determines the essential competition between two degrees of freedom and provides driving forces for phase transitions to each other or to other disordered phases. Dual confinement means that there is no true spin-charge separation is present in these ordered phases since one species (spinon/holon) is always confined at low temperatures while the other (spinon/holon) is condensed. But the dual deconfinement will play an essential role in the transitions to disordered phases or at the boundary between two ordered phases where a quantum critical point may exist[28,49]. The systematic evolution of the phase diagram at low doping is currently under investigation based on the mutual-Chern-Simons Lagrangian.

In the future we will also consider some additional relevant terms which have not been taken into consideration in the present *minimal* model. As previously shown[47], there generally exists a residual attractive interaction between holons and spinons within the $t-J$ model, which should be included when one considers the nodal (d-wave) fermionic quasiparticle excitations as "collective" modes in the SC phase. In principle, besides spinon confinement, there also exists a holon-spinon confinement in the SC phase of the phase string model, since single spinon or holon excitation is not allowed[47]. How the fermionic nodal quasiparticles can be naturally described in the mutual Chern-Simons framework will be a central issue to address in a next study, where the attractive interaction between holons and spinons beyond the phase string model should be properly incorporated in order to get a correct excitation spectrum[47]. We do not expect a qualitative modification on the present results of the minimal model by including such a term, since the quasiparticles as bound states of holons and spinons are independent, to leading order approximation, of those spinon excitations which are confined to form integer neutral spin excitaions discussed in the present work.

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APPENDIX A: SINGLE-VALUENESS OF $\Phi^h(\mathbf{r},\tau)$

In the transformation (26), $\Phi^h(\mathbf{r},\tau)$ is required to be single-valued with mod 2π in order to ensure the single-valueness of the spin operators. Then according to (27), it imposes a constraint on A^h_μ, i.e.,

$$\Delta\Phi^h|_C = 2\oint_C A^h_\mu dx_\mu = 2n\pi,\ n\in\mathbb{Z} \tag{A1}$$

in which dx_μ is the tangential differential vector of an arbitrary loop C in (2+1) dimensions.

Since the gauge field A^h_μ is an independent dynamic variable, the constraint (A1) would be generally *violated*. However, we shall prove below that all the A^h_μ configurations that violate (A1) have vanishing contribution to the partition function, which is consistent to the topological constraint (6) in the Hamiltonian formalism.

First of all, for an arbitrary loop C in (A1), we may introduce a vortex ring phase configuration $e^{i\theta(x_\mu)}$ with an arbitrary winding number M ($M\in\mathbb{Z}$), which satisfies $\oint_D d\theta = 2\pi M$ for any circuit D that winds around C once, as shown in Fig. 2. This singularity in $e^{i\theta(x,t)}$ can be clearly expressed by

$$\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta(x) = 2\pi M\oint_C dy_\mu(C)\delta^{(3)}\left(x_\mu - y_\mu(C)\right),\ M\in\mathbb{Z} \tag{A2}$$

in which $y_\mu(C)$ represents the coordinates on the loop C.

Then, we can make a singular gauge transformation in terms of such a phase $\theta(x_\mu)$ as

$$\widetilde{h} = e^{i\theta}h,\ \widetilde{A}^s_\mu = A^s_\mu + \partial_\mu\theta. \tag{A3}$$

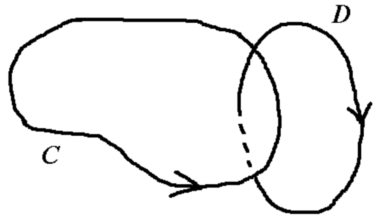

FIG. 2: A vortex ring phase θ is defined such that $\oint_D d\theta = 2\pi M$ for any circuit D winding around the loop C once.

Lagrangian $\mathcal{L}_h$ and $\mathcal{L}_s$ remain invariant, but the mutual-Chern-Simons term in (20) changes as

$$\widetilde{\mathcal{L}}_{CS} = \mathcal{L}_{CS} + \frac{i}{\pi}\partial_\mu \theta \epsilon^{\mu\nu\lambda}\partial_\nu A^h_\lambda \tag{A4}$$

such that the total action is transformed as

$$\begin{aligned} \widetilde{S}_{\text{eff}} &= S_{\text{eff}} + \int d^3x_\mu \frac{i}{\pi}\partial_\mu\theta\epsilon^{\mu\nu\lambda}\partial_\nu A^h_\lambda \\ &= S_{\text{eff}} + \int d^3x_\mu \frac{i}{\pi} A^h_\mu \epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta \\ &= S_{\text{eff}} + i2M\oint_C A^h_\mu dx_\mu . \end{aligned} \tag{A5}$$

Therefore, the partition function can be written as

$$\begin{aligned} Z &= \int D[..]\exp\left(-S_{\text{eff}} - 2iM\oint_C A^h_\mu dx_\mu\right) \\ &= Const\cdot\sum_{M\in\mathbb{Z}}\int D[..]\exp\left(-S_{\text{eff}} - iM\oint_C 2A^h_\mu dx_\mu\right) \end{aligned} \tag{A6}$$

where $D[..]$ stands for the functional integrations over all the fields, h, h^*, z_σ, z^*_σ, A^h_μ, and A^s_μ. The summation over M directly lead to the constraint (A1) for an arbitrary loop C.

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